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Laminar film condensation due to a rotating disk

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SUMMARY
Regular perturbation methods are employed to investigate the problem of laminar film condensation on a rotating disk in a large volume of quiescent vapour for small and large rates of cooling at the disk surface. The flow and thermal fields are represented by the Von Karman similar solution. Exact numerical solutions of the similar equations are obtained for the case of steam-water condensation and compared with those derived analytically.

1. Introduction

The work of Nusselt [1] on laminar film condensation on to a vertical flat plate has initiated many theoretical investigations into gravity driven and forced flows. Thus the boundary layer approximation was applied to Nusselt's problem by Sparrow and Gregg [2] and Koh, Sparrow and Hartnett [3] so as to include the effects of vapour drag and inertia and convective effects in the condensate flow. In [4] Cess formulated, in terms of the Blasius similar variables, the problem of forced vapour flow past a semi-infinite flat plate. The effect of variation of the physical properties of condensate with temperature in Nusselt's problem has been considered by Poots and Miles [5]. In all of the above theoretical investigations the essential mathematical difficulty lies in the solution of a system of nonlinear partial differential equations in two domains with an unknown boundary. This somewhat intractable numerical problem can be avoided on using perturbation methods. For example regular perturbation methods are shown by Beckett and Poots [6] to yield reliable quantitative results for plane two-dimensional forced flows even when the vapour pressure gradient is adverse.

The simplicity of the Nusselt model for laminar film condensation motivated Sparrow and Gregg [7] to investigate the case of a rotating disk in a large volume of quiescent vapour. They assumed a constant property condensate whose flow and thermal fields were represented by the well known Von Karman similar solution. In particular they neglect the effect of vapour drag on assuming a stress-free condensate-vapour interface. No subsequent results are known to have been published on this topic and although a recent contribution by Dhir and Lienhard [8] discusses the disk problem no new information is given.

Because of this lack of quantitative and qualitative theoretical information on condensation in rotating flows the case of a rotating disk is re-examined. The regular perturbation procedures developed in [6] for small and large rates of cooling at the disk surface are employed. Exact numerical solutions of the relevant Von Karman similar flow field are obtained for the case of steam-water condensation. This information is then employed to assess the accuracy of the analytical predictions.

The flow configuration is as follows: An infinite disk at temperature T_w rotates with angular velocity Ω in a large volume of quiescent pure vapour at saturation temperature $T_s > T_w$. A condensate film is formed on the disk and a steady state axially symmetric flow exists in which both vapour and condensate are swept radially outwards in spiral paths away from the axis of rotation. The steady renewal of condensate maintains the interface in a fixed position. The actual location being dependent on the scale of rotation and the rate of cooling at the disk surface.

Liquid Phase: In the liquid phase cylindrical polar coordinates (r, ϕ, z) are used, r being the distance measured from the axis of rotation, ϕ the polar angle measured in the direction of rotation and z the normal distance from the disk. The velocity field has components (u, v, w) in the (r, ϕ, z)-directions and the temperature field is denoted by T.

The Von Karman similar solution, see Ostrach and Thorton [9], is given in the form:

$$\left.\begin{aligned} r &= \left(\frac{\mu}{\rho\Omega}\right)_s^{\frac{1}{2}} R\,, \quad z = \left(\frac{\mu}{\rho\Omega}\right)_s^{\frac{1}{2}} Z\,, \quad p = \mu_s \Omega P(Z)\,, \quad \theta(Z) = \frac{T - T_w}{T_s - T_w}\,, \\ u &= \left(\frac{\Omega\mu}{\rho}\right)_s^{\frac{1}{2}} Rf(Z)\,, \quad v = \left(\frac{\Omega\mu}{\rho}\right)_s^{\frac{1}{2}} Rg(Z)\,, \quad w = \left(\frac{\Omega\mu}{\rho}\right)_s H(Z)\,. \end{aligned}\right\} \tag{1}$$

Here μ denotes the viscosity, ρ the density, K the thermal conductivity, c_p the specific heat at constant pressure and the subscript s the condition at saturation temperature T_s. Except for the R-dependence in the radial and angular velocity components it is observed that the velocity and thermal fields are functions of Z. Consequently the thickness of the condensate layer is constant and independent of R; the liquid-vapour interface is located at

$$Z = \Delta = \text{const}\,. \tag{2}$$

Further simplification is achieved on employing the Howarth–Dorodnitsyn variable

$$x = \frac{1}{\Phi}\int_0^Z \frac{\rho}{\rho_s}\, dZ\,, \quad \Phi = \int_0^\Delta \frac{\rho}{\rho_s}\, dZ\,; \tag{3}$$

moreover the inflow velocity function $H(Z)$ is written as follows

$$H(Z) = \frac{\rho_s}{\rho}\, h(x)\,. \tag{4}$$

The range of integration is now $0 \leqq x \leqq 1$. The Navier–Stokes equations expressing conservation of mass, momentum and thermal energy become the ordinary differential equations:

$$\left.\begin{aligned} & h' + 2\Phi f = 0\,, \quad \left(\frac{\mu\rho}{(\mu\rho)_s} f'\right)' = \Phi^2(f^2 - g^2) + \Phi h f'\,, \\ & \left(\frac{\mu\rho}{(\mu\rho)_s} g'\right)' = 2\Phi^2 f g + \Phi h g'\,, \quad \left(\frac{K\rho}{(K\rho)_s}\theta'\right)' = \frac{c_p}{c_{p_s}}\Phi P_s h\theta'\,, \end{aligned}\right\} \tag{5}$$

where the dash denotes differentiation with respect to x. The equations are supplemented with variable fluid property relations

$$\rho = \rho(T)\,, \quad c_p = c_p(T)\,, \quad \mu = \mu(T) \quad \text{and} \quad K = K(T)\,. \tag{6}$$

It is assumed that this condensate data is available from experiment (see [10]).

Vapour Phase: For convenience all vapour quantities are denoted by a star. For example (u^*, v^*, w^*) denotes the velocity components in the (r, ϕ, z^*)-directions, where z^* now measures the normal distance from the liquid-vapour interface. The isothermal flow of the vapour at saturation temperature T_s is now given by the similar solution:

$$\left.\begin{aligned} z^* &= \left(\frac{\mu^*}{\rho^*\Omega}\right)_s^{\frac{1}{2}} x^*\,, \quad p^* = \mu_s^* \Omega P^*(x^*)\,, \\ u^* &= \left(\frac{\Omega\mu^*}{\rho^*}\right)_s^{\frac{1}{2}} Rf^*(x^*)\,, \quad v^* = \left(\frac{\Omega\mu^*}{\rho^*}\right)_s^{\frac{1}{2}} Rg^*(x^*)\,, \\ w^* &= \left(\frac{\Omega\mu^*}{\rho^*}\right)_s^{\frac{1}{2}} h^*(x^*)\,. \end{aligned}\right\} \tag{7}$$

The velocity functions f^*, g^* and h^* satisfy the differential equations:

$$h^{*\prime}+2f^*=0\,,\quad f^{*\prime\prime}=f^{*2}-g^{*2}+h^*f^{*\prime}\,,\quad g^{*\prime\prime}=2f^*g^*+h^*g^{*\prime}\,. \tag{8}$$

Boundary conditions: At the surface of the disk the usual conditions of no slip and no temperature jump are required, i.e.

$$u=0\,,\quad v=r\Omega\,,\quad w=0\,,\quad T=T_w\,. \tag{9}$$

As the disk is rotating in a large volume of quiescent vapour then

$$u^*\to 0\,,\quad v^*\to 0 \text{ as } z^*\to\infty\,. \tag{10}$$

It is also necessary to match the condensate and vapour flows at the interface located at $x=1$ or $x^*=0$. The necessary conditions are:

(i) Continuity in mass flow, yielding

$$\rho_s w=\rho_s^* w^*\,; \tag{11}$$

(ii) Continuity in the tangential components of velocity, yielding

$$u=u^*\,,\quad v=v^*\,; \tag{12}$$

(iii) Continuity in shear stress, yielding

$$\widehat{z\phi}=\widehat{z^*\phi^*}^*,\quad \widehat{rz}=\widehat{rz}^{**},\quad \widehat{zz}=\widehat{z^*z^*}^*, \tag{13}$$

where the stresses are given by

$$\widehat{rz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right),\quad \widehat{z\phi}=\mu\frac{\partial v}{\partial z},\quad \widehat{zz}=-p+2\mu\frac{\partial w}{\partial z}\,.$$

(iv) Finally at the interface $Z=\Delta$

$$T=T_s \quad\text{and}\quad K_s\left(\frac{\partial T}{\partial Z}\right)_\Delta=-\rho_s^* w^* h_{fg}\,, \tag{14}$$

where h_{fg} is the latent heat of condensation.

In terms of the similar variables the boundary conditions for determining f, g, h, θ and f^*, g^*, h^* and the unknown dimensionless thickness Φ are:

$$\text{At}\quad x=0:\quad f=h=0\,,\ g=1\,,\ \theta=0\,; \tag{15}$$

At $x=1$ and $x^*=0$:

$$\left.\begin{aligned}&h^*(0)=\lambda h(1)\,,\ f^*(0)=f(1)\,,\ g^*(0)=g(1)\,,\ \Phi f^{*\prime}(0)=\lambda f'(1)\,,\\&\Phi g^{*\prime}(0)=\lambda g'(1)\,,\ \theta(1)=1 \text{ and } \chi\theta'(1)=-\Phi h(1)\,.\end{aligned}\right\} \tag{16}$$

$$\text{As}\quad x^*\to\infty:\ f^*\to 0\,,\ g^*\to 0\,. \tag{17}$$

Here $\lambda=(\nu/\nu^*)_s^{\frac{1}{2}}(\rho/\rho^*)_s$, $\chi=c_p\Delta T/P_s h_{fg}$, the Prandtl number $P=c_p\mu/K$, and $\nu=\mu/\rho$ is the kinematic viscosity.

The quantities of practical interest are as follows.

Torque and Moment Coefficient: The torque M on one side of a disk of radius R_0 is

$$M=-2\pi\int_0^{R_0} r^2\,\widehat{z\phi}\,dr\,,\quad \widehat{z\phi}=\mu_s\left(\frac{\partial v}{\partial z}\right)_{z=0}\,.$$

The dimensionless moment coefficient C_M is

$$C_M=M/\tfrac{1}{2}\rho_s\Omega^2R_0^5=-2\pi\,\mathrm{Re}^{-\frac{1}{2}}\frac{(\rho\mu)_w}{(\rho\mu)_s}\frac{g'(0)}{\Phi}\,, \tag{18}$$

where the Reynolds number $\mathrm{Re}=\rho_s\Omega R_0^2/\mu_s$.

Nusselt Number: This is given by

$$\mathrm{Nu} = R_0 \left(\frac{\partial T}{\partial z}\right)_{z=0} \bigg/ \Delta T = \mathrm{Re}^{\frac{1}{2}} \frac{(\rho K)_w}{(\rho K)_s} \frac{\theta'(0)}{\Phi} . \tag{19}$$

Condensate Film Thickness: If δ is the actual condensate thickness then from (1)–(3) it follows

$$\frac{\delta}{R_0} = \mathrm{Re}^{-\frac{1}{2}} \Phi \int_0^1 \frac{\rho_s}{\rho} dx . \tag{20}$$

The system of equations (5) and (6) for the condensate flow and (8) for the vapour flow subject to the boundary conditions (15)–(17) can be solved by numerical methods. For example a matrix interpretative scheme employing quasi-linearisation proved to be satisfactory.

In Table 1 some numerical results are listed for steam-water condensation for pure steam at saturation temperature 100°C and at atmospheric pressure, the disk temperatures being $T_w = 0(20)$ 80°C and 90°C. In Figures 1 and 2 representative velocity profiles are displayed for $T_w = 0$°C and 90°C.

To estimate the magnitude of the errors introduced on neglecting the effects of vapour drag, as in the work of Sparrow and Gregg [7], a stress-free interface is assumed. Then it is necessary to solve the condensate equations (5) and (6) subject to the conditions (15) at $x=0$, and to replace (16) at $x=1$ with the following:

$$f''(1) = g'(1) = 0 , \quad \theta(1) = 1 \text{ and } \chi\theta'(1) = -\Phi h(1) . \tag{21}$$

TABLE 1

Full numerical solution for steam-water condensation

T_w °C	$\theta'(0)$	$f''(0)$	$g'(0)$	Φ	$C_M \mathrm{Re}^{\frac{1}{2}}/2\pi$	$\mathrm{Nu}\,\mathrm{Re}^{-\frac{1}{2}}$
0	1.156	0.0994	−0.0285	0.847	0.223	1.150
20	1.085	0.140	−0.0417	0.753	0.206	1.317
40	1.043	0.167	−0.0496	0.661	0.181	1.518
60	1.017	0.175	−0.0479	0.565	0.121	1.785
80	1.006	0.151	−0.0328	0.449	0.0835	2.245
90	1.005	0.117	−0.0189	0.367	0.0581	2.746

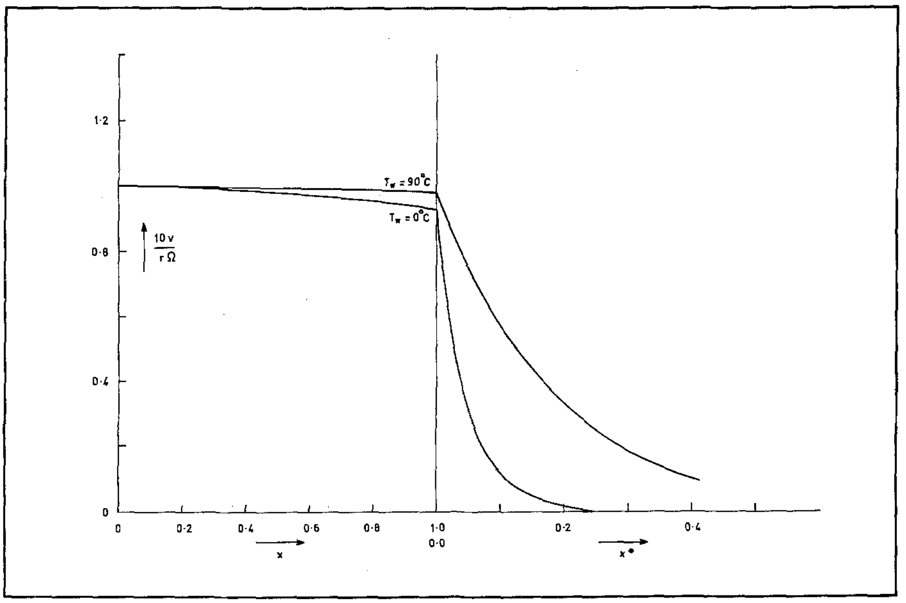

Figure 1. Dimensionless radial velocity profiles for $T_w = 0$ and 90°C.

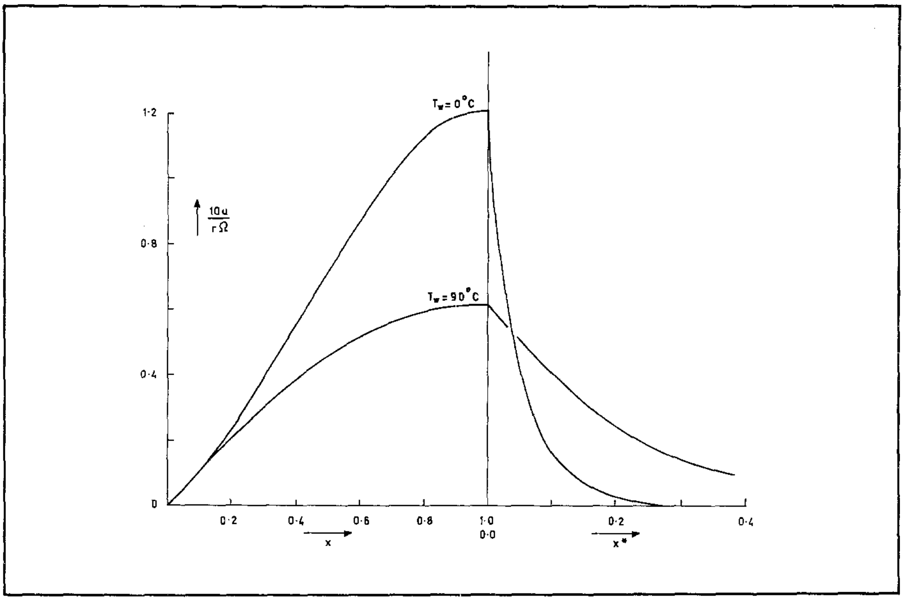

Figure 2. Dimensionless angular velocity for $T_w = 0$ and 90°C.

TABLE 2

Numerical solution with stress-free interface

T_w °C	$\theta'(0)$	$f'(0)$	$g'(0)$	Φ	$C_M \mathrm{Re}^{\frac{1}{2}}/2\pi$
0	1.155	0.100	−0.0147	0.828	0.118
20	1.085	0.142	−0.0216	0.737	0.109
40	1.043	0.169	−0.0255	0.639	0.0950
60	1.017	0.177	−0.0245	0.556	0.0628
80	1.006	0.152	−0.0166	0.445	0.0478
90	1.005	0.117	−0.0095	0.365	0.0294

The results for steam-water condensation are given in Table 2. When compared with those listed in Table 1 excellent agreement can be noted for the characteristics Φ, $\theta'(0)$ and $f'(0)$ but, as might be expected, $g'(0)$ is in error by 50 per cent. Clearly the omission of interfacial shear does not lead to errors in condensate thickness or the surface heat transfer coefficient but gives serious discrepancies in the moment coefficient.

2. Perturbation methods of solution

As shown by Beckett and Poots [6] there are two limiting forms for the condensation process. These correspond to the physical situations of small or large rates of cooling at the disk surface. In the following uniformly valid expansions are derived for these limiting conditions. The perturbation methods depend on the dimensionless groups $\chi = c_p \Delta T / P_s h_{fg}$ and $\lambda = (v/v^*)_s^{\frac{1}{2}} (\rho/\rho^*)_s$. For most vapours employed in engineering practice $\chi \ll 1$ and $\lambda \gg 1$; for example in the case of steam-water condensation $\lambda = 191$ and $\chi \leqq \frac{1}{10}$ for $0 \leqq T_w \leqq 100$°C.

2.1. *The Thin Film Approximation*

In the limit as $\Delta T = (T_s - T_w) \to 0$ or $\chi \to 0$ the dimensionless condensate thickness $\Phi \to 0$ and the vapour velocity functions f^*, g^* and h^* are those pertaining to the single phase vapour flow on a uniformly rotating impermeable disk. Moreover in the limiting process $\chi \to 0$ the condensate behaves as a constant property fluid; for example $\mu_w - \mu_s \sim \chi$ as $\chi \to 0$.

The governing equations (5) for the condensate simplify to the following

$$\left.\begin{aligned} &h'+2\Phi f=0\,, \quad f''=\Phi^2(f^2-g^2)+\Phi hf'\,,\\ &g''=2\Phi^2 fg+\Phi hg'\,, \quad \theta''=\Phi P_s h\theta' \end{aligned}\right\} \tag{21}$$

which coupled with the vapour phase equations (8), are to be solved subject to the boundary conditions (15)–(17).

To satisfy the limiting requirements as $\chi\to 0$ it is necessary to expand the velocity and thermal fields in terms of the parameter $\varepsilon=\chi^{\frac{1}{3}}$. For clearly if $\Phi=O(\chi^m)$ and $h=O(\chi^n)$ then from the equation of continuity $f=O(\chi^{n-m})$ and from the stress conditions and heat balance at the interface it follows that $2m-n=0$ and $m+n=1$, respectively. Thus $\Phi\sim\chi^{\frac{1}{3}}$, $h\sim\chi^{\frac{2}{3}}$ and $f\sim\chi^{\frac{1}{3}}$ as $\chi\to 0$.

The regular perturbation expansions* are:

$$\left.\begin{aligned} \text{Condensate: } \Phi &= \sum_1^\infty \varepsilon^n\phi_n\,, \quad \theta=\sum_0^\infty \varepsilon^n\theta_n(x)\,, \quad f=\sum_1^\infty \varepsilon^n f_n(x)\,,\\ g &= \sum_0^\infty \varepsilon^n g_n(x)\,, \quad h=\sum_2^\infty \varepsilon^n h_n(x)\,; \end{aligned}\right\} \tag{22}$$

$$\text{Vapour:} \quad f^*=\sum_0^\infty \varepsilon^n f_n^*(x^*)\,, \quad g^*=\sum_0^\infty \varepsilon^n g_n^*(x^*)\,, \quad h^*=\sum_0^\infty \varepsilon^n h_n^*(x^*)\,, \tag{23}$$

The perturbation equations are as follows:

Zeroth-order flow:

$$h_0^{*\prime}+2f_0^*=0\,, \quad f_0^{*\prime\prime}=f_0^{*2}-g_0^{*2}+h_0^*f_0^{*\prime}\,, \quad g_0^{*\prime\prime}=2f_0^*g_0^*+h_0^*g_0^{*\prime}\,; \tag{24}$$

$$f_1''=g_0''=\theta_0''=0\,, \quad h_2'=-2\phi_1 f_1\,. \tag{25}$$

the boundary conditions are

$$\begin{aligned} &f_1(0)=h_2(0)=0\,, \; g_0(0)=1\,, \; f_0^*(\infty)=g_0^*(\infty)=0\,, \; \theta_0(0)=0\,, \; \theta_0(1)=1\,,\\ &g_0(1)=g_0^*(0),\; f_0^*(0)=0,\; h_0^*(0)=0,\; f_1'(1)=\frac{\phi_1}{\lambda}f_0^{*\prime}(0),\; g_0'(1)=0,\; \theta_0'(1)=-\phi_1 h_2(1)\,. \end{aligned} \tag{26}$$

The vapour and condensate equations are now uncoupled. The solution to the system (24) and (26) is known since it governs the Von Karman similar solution for the flow due to an impermeable rotating disk. The characteristics, given in ref. [9] are

$$f_0^{*\prime}(0)=A_1=0.510\,, \; g_0^{*\prime}(0)=B_1=-0.616\,, \; h_0^*(\infty)=-0.886\,. \tag{27}$$

The solution of (25) and (26) yields

$$\theta_0(x)=x\,, \; g_0(x)=1\,, \; f_1(x)=A_1\phi_1 x/\lambda\,, \; h_2(x)=-\phi_1^2 A_1\frac{x^2}{\lambda}\,, \; \phi_1=\left(\frac{\lambda}{A_1}\right)^{\frac{1}{3}}. \tag{28}$$

Thus the zeroth order condensate flow is activated by the $\widehat{rz}^{**}$stress but is in a state of rigid body rotation.

First-order flow: For the vapour it is necessary to solve a linear system of equations. Set

$$h_1^*=\left(\frac{A_1}{\lambda}\right)^{\frac{2}{3}} H_1^*(x^*)\,, \; f_1^*=\left(\frac{A_1}{\lambda}\right)^{\frac{2}{3}} F_1^*(x^*)\,, \; g_1^*=\left(\frac{A_1}{\lambda}\right)^{\frac{2}{3}} G_1^*(x^*)\,, \tag{29}$$

where H_1^*, F_1^* and G_1^* satisfy

* As previously noted $\mu_w-\mu_s=O(\varepsilon^3)$ and the assumption of a constant property model is consistent provided third order terms are not required.

$$\left.\begin{array}{l} H_1^{*\prime}+2F_1^*=0\,,\ F_1^{*\prime\prime}=2f_0^*F_1^*-2g_0^*G_1^*+h_0^*F_1^{*\prime}+H_1^*f_0^{*\prime}\,,\\ G_1^{*\prime\prime}=2f_0^*G_1^*+2g_0^*F_1^*+h_0^*G_1^{*\prime}+H_1^*g_0^{*\prime}\,, \end{array}\right\} \tag{30}$$

subject to the boundary conditions

$$F_1^*(0)=1\,,\ G_1^*(0)=B_1/A_1\,,\ H_1^*(0)=0\,,\ F_1^*(\infty)=0\,,\ G_1^*(\infty)=0\,. \tag{31}$$

The solution of (30) and (31) is clearly $F_1^*=f_0^{*\prime}/A_1$, $G_1^*=g_0^{*\prime}/A_1$ and $H_1^*=h_0^{*\prime}/A_1$. This yields the characteristics

$$F_1^{*\prime}(0)=A_2=-1/A_1\,,\quad G_1^{*\prime}(0)=B_2=0\,, \tag{32}$$

which are required to determine the condensate flow. This is now given by the expressions

$$\left.\begin{array}{l} \theta_1(x)=0\,,\ g_1(x)=\dfrac{\phi_1 B_1}{\lambda}x\,,\ \phi_2=-\tfrac{1}{3}\left(\dfrac{\lambda}{A_1}\right)^{\frac{2}{3}}\left\{\dfrac{2\lambda}{3A_1}+\dfrac{A_2}{\lambda}\right\},\\ f_2(x)=\phi_1^2\left(x-\dfrac{x^2}{2}\right)+\dfrac{x}{\lambda}\left\{A_2\left(\dfrac{A_1}{\lambda}\right)^{\frac{1}{3}}+A_1\phi_2\right\},\\ h_3(x)=\phi_1^3\left(\dfrac{x^3}{3}-x^2\right)+\dfrac{\phi_1}{\lambda}x^2\left\{A_2\left(\dfrac{A_1}{\lambda}\right)^{\frac{1}{3}}+2A_1\phi_2\right\} \end{array}\right\} \tag{33}$$

On the basis of the thin film approximation formulae are now available for the flow characteristics C_M, Nu and δ as previously defined. The first two terms in the expansion yield:

$$\left.\begin{array}{l} \Phi=\left(\dfrac{\lambda\chi}{A_1}\right)^{\frac{1}{3}}-\tfrac{1}{3}\left(\dfrac{\lambda\chi}{A_1}\right)^{\frac{2}{3}}\left(\dfrac{2\lambda}{3A_1}+\dfrac{A_2}{\lambda}\right)+\ldots,\\ g'(0)=\left(\dfrac{\lambda\chi}{A_1}\right)^{\frac{1}{3}}\dfrac{B_1}{\lambda}+\ldots,\quad \theta'(0)=1+\ldots. \end{array}\right\} \tag{34}$$

It is clear from (34) that the convergence rate is poor. For in the case of steam-water condensation, for which $\lambda=191$, and $\chi=1.07\times10^{-3}\,\Delta T/°\mathrm{C}$, the ratio of the first order correction in Φ to the zeroth-order contribution yields the inequality $\Delta T<10^{-5}\,°\mathrm{C}$.

Thus the thin film approximation, although being a uniformly valid expansion in the limit $\chi\to0$, is of little practical value in the study of condensation in rotating flows. It should be emphasized, however, that such flows have a general feature namely that in some part of the flow field vapour is pulled towards the solid rotating surface. Thus even for small ΔT the condensation rate is probably significant. In the next section the concept of a suction boundary layer at the interface is utilized to develop a perturbation analysis for large condensation rates.

2.2. *The Thick Film Approximation*

For $\Delta T/T_s=O(1)$ the rate of condensation will be appreciable. The liquid-vapour interface is considered to be a fictitious rotating disk with angular velocity less than Ω. The condensate is then in a state of near rigid body rotation and the vapour flow is controlled by strong suction at the fictitious disk (or interface).

Since λ is large introduce new suction boundary layer variables for the vapour flow, namely

$$X^*=\lambda x^*\,,\ h^*=\lambda H^*(x^*)\,,\ f^*=\lambda^2F^*(x^*)\,,\ g^*=\lambda^2G^*(x^*)\,. \tag{35}$$

On substitution of (35) into (8) suitable asymptotic expansions for H_1^*, F^* and G^* are (see Stuart [11]):

$$H^*=-S^*+\frac{1}{\lambda^2}H_2^*(x^*)+\ldots,\ F^*=\frac{1}{\lambda^2}F_2^*(x^*)+\ldots,\ G^*=\frac{1}{\lambda^2}G_2^*(x^*)+\ldots, \tag{36}$$

where the unknown suction velocity at the interface is denoted by S^*. Due to the far flow field conditions it follows that

$$F_2^* = B_2^* \exp(-S^* X^*) \quad \text{and} \quad G_2^* = C_2^* \exp(-S^* X^*) .$$

The unknown constants B_2^*, C_2^*, S^* are now determined on matching the vapour suction boundary layer to the condensate flow at the interface. To achieve this the liquid phase expansion must be of the form:

$$\left. \begin{aligned} & f = F + O\left(\frac{1}{\lambda}\right), \quad g = G + O\left(\frac{1}{\lambda}\right), \quad h = H + O\left(\frac{1}{\lambda}\right), \\ & \theta = \Theta + O\left(\frac{1}{\lambda}\right), \quad \Phi = \tilde{\Phi} + O\left(\frac{1}{\lambda}\right), \end{aligned} \right\} \tag{37}$$

together with similar expansions for ρ, μ, c_p and K. The zeroth-order functions F etc. now satisfy the system of differential equations (5) with f replaced by F etc.

The boundary conditions at the interface now become

$$\left. \begin{aligned} & F(1) = F_2^*(0) = B_2^* , \; G(1) = G_2^*(0) = C_2^* , \; H(1) = -S^* , \\ & \tilde{\Phi} F_2^{*\prime}(0) = F'(1) , \; \tilde{\Phi} G_2^{*\prime}(0) = G'(1) \text{ and } \chi \Theta'(1) = -\tilde{\Phi} H(1) , \end{aligned} \right\} \tag{38}$$

from which C_2^*, B_2^* and S^* may be eliminated. Consequently in the limit as $\lambda \to \infty$ the boundary conditions are:

$$\left. \begin{aligned} & F = H = \Theta = 0 , \; G = 1 \text{ at } x = 0 ; \\ & \Theta = 1 , \; F' = \tilde{\Phi} F H , \; G' = \tilde{\Phi} G H , \; \chi \Theta' = -\tilde{\Phi} H \text{ at } x = 1 . \end{aligned} \right\} \tag{39}$$

The solution of (5) subject to (39), with the full variable fluid properties, has been found numerically for steam-water condensation. These new results are not listed as they are identical (to three significant figures) with those in Table 1. Hence, the thick film approximation is in precise agreement with the full numerical solutions throughout the range $0 \leqq T_w \leqq 90°$C. The real advantage gained is the elimination of the vapour phase flow equations.

The remaining question to be resolved is the limiting form of the thick film approximation as $\chi \to 0$. Clearly these equations will become invalid since $S^* \to 0$ with $\chi \to 0$. This situation is already covered by the thin film approximation.

The behaviour for small χ of the zeroth-order condensate functions is found to be

$$\tilde{\Phi} = \chi^{\frac{1}{4}} \tilde{\phi} , \; H = \chi^{\frac{3}{4}} \tilde{h} , \; F = \chi^{\frac{1}{2}} \tilde{f} , \; G = 1 + \chi \tilde{G} , \; \Theta = \tilde{\theta} . \tag{40}$$

On substituting these into (5) (with f replaced by F etc.) and (39) the equations governing $\tilde{f}$ etc. are as follows:

$$\left. \begin{aligned} & \tilde{h}' + 2\tilde{\phi}\tilde{f} = 0 , \\ & \left(\frac{\rho\mu}{\rho_s\mu_s} \tilde{f}'\right)' + \tilde{\phi}^2 = \chi[\tilde{\phi}^2(\tilde{f}^2 - 2\tilde{g}) + \tilde{h}\tilde{\phi}\tilde{f}'] - \chi^2 \tilde{g}^2 \tilde{\phi}^2 , \\ & \left(\frac{\rho\mu}{\rho_s\mu_s} \tilde{g}'\right)' - 2\tilde{f}\tilde{\phi}^2 = \chi[2\tilde{f}\tilde{\phi}^2\tilde{g} + \tilde{\phi}\tilde{h}\tilde{g}'] , \\ & \left(\frac{\rho K}{\rho_s K_s} \tilde{\theta}'\right)' = \chi P_s \frac{c_p}{c_{p_s}} \tilde{\phi}\tilde{h}\tilde{\theta}' . \end{aligned} \right\} \tag{41}$$

The appropriate boundary conditions are:

$$\left. \begin{aligned} & \tilde{f}(0) = \tilde{g}(0) = \tilde{h}(0) = \tilde{\theta}(0) = 0 , \; \tilde{\theta}(1) = 1 , \; \tilde{\theta}'(1) = -\tilde{\phi}\tilde{h}(1) \\ & \tilde{f}'(1) = \chi\tilde{\phi}\tilde{h}(1)\tilde{f}(1) , \; \tilde{g}'(1) = \tilde{\phi}\tilde{h}(1)(1 + \chi\tilde{g}(1)) . \end{aligned} \right\} \tag{42}$$

To find the solution of this set of equations (41) and (42) for small χ the variables are expanded as follows:

$$\left.\begin{aligned} &\tilde{h}(x)=h_0(\tau)+\chi h_1(\tau)+O(\chi^2)\,,\ \tilde{f}(x)=f_0(\tau)+\chi f_1(\tau)+O(\chi^2)\,,\\ &\tilde{g}(x)=g_0(\tau)+\chi g_1(\tau)+O(\chi^2)\,,\ \tilde{\Phi}=\phi_0+\chi\phi_1+O(\chi^2)\,,\\ \text{and}\quad &\tilde{\theta}(x)=\tau+\chi\theta_1(\tau)+\chi^2\theta_2(\tau)+O(\chi^3)\,, \end{aligned}\right\} \tag{43}$$

where τ is the independent variable, $0 \leqq \tau \leqq 1$. Expansions are also required for ρ, c_p, μ and K. However to avoid algebraic complexity it is informative to introduce a model fluid for the condensate with properties:

$$\rho=\rho_s\,,\ c_p=c_{p_s}\,,\ K=K_w+(K_s-K_w)\tilde{\theta}\,,\ \frac{1}{\mu}=\frac{1}{\mu_w}+\left(\frac{1}{\mu_s}-\frac{1}{\mu_w}\right)\tilde{\theta}\,. \tag{44}$$

It is relatively easy to show that the zeroth-order functions are then governed by the system of equations:

$$\begin{aligned} &E^2\frac{K_s}{K_0}\left\{\frac{\mu_0 K_s}{\mu_s K_0}f_0'\right\}'+\phi_0^2=0\,,\quad E\frac{K_s}{K_0}h_0'+2\phi_0 f_0=0\,,\\ &E^2\frac{K_s}{K_0}\left\{\frac{\mu_0 K_s}{\mu_s K_0}g_0'\right\}'-2\phi_0^2 f_0=0\,, \end{aligned} \tag{45}$$

subject to the boundary conditions

$$f_0(0)=g_0(0)=h_0(0)=0\,,\ f_0'(1)=0\,,\ \phi_0 h_0(1)=-E\ \text{and}\ g_0'(1)=-1\,. \tag{46}$$

Here the dash now denotes differentiation with respect to τ and E is a constant given by:

$$E=\int_0^1\frac{K_0}{K_s}\,d\tau=\frac{(K_s+K_w)}{2K_s}\,. \tag{47}$$

A solution of these equations, see Poots and Miles [5], yields the following formulae for the characteristics:

$$\Phi=\left(\frac{3\chi}{2}\right)^{\frac{1}{4}}\frac{E}{\varepsilon_p}+\dots,\quad \frac{C_M \mathrm{Re}^{\frac{1}{2}}}{2\pi}=\frac{2\chi E}{\Phi}+\dots,\quad \mathrm{Nu}\,\mathrm{Re}^{-\frac{1}{2}}=E/\Phi+\dots, \tag{48}$$

where

$$\varepsilon_p=\left[\frac{1}{280}\left(16+29\frac{K_w}{K_s}+20\left(\frac{K_w}{K_s}\right)^2+5\left(\frac{K_w}{K_s}\right)^3+\right.\right.$$
$$\left.\left.+\frac{\mu_s}{\mu_w}\left\{19+76\frac{K_w}{K_s}+85\left(\frac{K_w}{K_s}\right)^2+30\left(\frac{K_w}{K_s}\right)^3\right\}\right)\right]^{\frac{1}{4}}. \tag{49}$$

The information predicted by these, together with $(df/dz)_{z=0}$, $(dg/dz)_{z=0}$ and $(dh/dz)_{z=0}$ are given in Table 3. It is evident that the heat transfer and torque are in good agreement with the exact numerical solutions. This fortuitious agreement is due to algebraic cancellation of the combined errors in the zeroth-order functions. Thus it is desirable to examine the next approximation.

The first-order functions satisfy the equations

TABLE 3

Zeroth order contributions (48)

T_w °C	$\theta'(0)$	$f'(0)$	$g'(0)$	ϕ	$C_M\,\mathrm{Re}^{\frac{1}{2}}/2\pi$	$\mathrm{Nu}\,\mathrm{Re}^{-\frac{1}{2}}$
0	1.120	0.0974	−0.0307	0.786	0.247	1.149
20	1.070	0.143	−0.0454	0.714	0.227	1.315
40	1.039	0.173	−0.0537	0.635	0.196	1.159
60	1.018	0.181	−0.0510	0.548	0.154	1.792
80	1.007	0.155	−0.0339	0.441	0.0969	2.250
90	1.002	0.118	−0.0192	0.364	0.0591	2.743

$$\left.\begin{aligned}
&E\frac{K_s}{K_0}h_1' = -2\phi_0 f_1 - 2\phi_1 f_0\,,\\
&E^2\frac{K_s}{K_0}\left[\frac{\mu_0 K_s}{\mu_s K_0}f_1'\right]' + E^2\frac{K_s}{K_0}\left[\frac{\mu_1 K_s}{\mu_s K_0}f_0'\right]' =\\
&\quad = -2\phi_0\phi_1 + \phi_0^2(f_0^2 - 2g_0) + E\frac{K_s}{K_0}\phi_0 h_0 f_0'\,,\\
&E^2\frac{K_s}{K_0}\left[\frac{\mu_0 K_s}{\mu_s K_0}g_1'\right]' + E^2\frac{K_s}{K_0}\left[\frac{\mu_1 K_s}{\mu_s K_0}g_0'\right]' =\\
&\quad = 2f_1\phi_0^2 + 4\phi_0\phi_1 f_0 + 2\phi_0^2 f_0 g_0 + E\frac{K_s}{K_0}\phi_0 h_0 g_0'\,,\\
&E\left[\theta_1' + \frac{K_1}{K_0}\right]' = P_s\phi_0 h_0\,;
\end{aligned}\right\} \tag{50}$$

the boundary conditions are:

$$\left.\begin{aligned}
&f_1(0) = g_1(0) = h_1(0) = \theta_1(0) = \theta_1(1) = 0\,,\\
&Ef_1'(1) = \phi_0 h_0(1) f_0(1)\,,\ E\theta_1'(1) = -\phi_0 h_1(1) - \phi_1 h_0(1)\,,\\
&Eg_1'(1) = -E\theta_1'(1) + g_0(1)\phi_0 h_0(1)\,.
\end{aligned}\right\} \tag{51}$$

Analytical solution of this set is straight forward but laborious and gives unwieldy formulae for the characteristics. These will not be given but instead actual numerical results are listed in Table 4 for steam-water condensation. It can be seen that when $T_w > 40°C$ inclusion of the first-order terms leads to improved results but as T_w approaches 0°C odd deviations appear. These are due to the marked non-linearity in μ and K, and the neglect of density variation, all of which combine to distort the exact solution in the vicinity of $T_w = 0°C$.

TABLE 4

Zeroth and first order contributions

T_w °C	$\theta'(0)$	$f'(0)$	$g'(0)$	Φ	$C_M\,\mathrm{Re}^{\frac{1}{2}}/2\pi$	$\mathrm{Nu}\,\mathrm{Re}^{-\frac{1}{2}}$
0	1.136	0.0924	−0.0277	0.806	0.218	1.137
20	1.084	0.136	−0.0414	0.732	0.202	1.300
40	1.049	0.166	−0.0497	0.650	0.178	1.500
60	1.025	0.175	−0.0481	0.559	0.143	1.770
80	1.010	0.151	−0.0329	0.447	0.0928	2.232
90	1.004	0.117	−0.0189	0.366	0.0578	2.731

Returning briefly to the discussion of the effect of the interfacial shear on the solution, the factor of 2 which appears in the numerical solution is clearly manifested in these new approximations. For if a stress-free interface condition is assumed the zeroth-order contribution to the moment coefficient is $C_M\mathrm{Re}^{\frac{1}{2}}/2\pi = \chi E/\varepsilon_p$ as compared with the result given in (48).

Finally it is of interest to examine the range of applicability of the thick film expansion in the region $\Delta T \sim 0°C$, where the thin-film expansion is valid. In the case steam-water condensation the zeroth-order contribution (48) gives $\Phi \sim \frac{1}{5}(\Delta T/°C)^{\frac{1}{4}}$. For $\Delta T = 1$, 10^{-2}, 10^{-5} °C this yields $\Phi = 0.200$, 0.0632 and 0.0112 which are in excellent agreement with computed exact values $\Phi = 0.210$, 0.0624 and 0.0104 respectively. Thus the region of validity of the thick film expansion extends to the outer region of the thin film expansion. Indeed for all practical purposes the formulae (48) are to be recommended.

3. Conclusions

The mechanism of laminar film condensation on a rotating disk is described completely by the analytical formulae for the thin and thick film approximations.

It is also possible to infer that the new analytical formulae presented are in better agreement with experiment than those proposed by Sparrow and Gregg [7]. There, the constant property analysis gives $Nu/Re^{\frac{1}{2}} = (2/3\chi)^{\frac{1}{4}}$ which yields results 30 per cent higher than experimental values. The result obtained here is $Nu/Re^{\frac{1}{2}} = \varepsilon_p(2/3\chi)^{\frac{1}{4}}$, where in the case of steam-water condensation ε_p takes values between 0.73 and 1 for $0 \leqq T_w \leqq 100\,^\circ\mathrm{C}$.

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